

# ON THE EXPANSIONS OF REAL NUMBERS IN TWO INTEGER BASES

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ABSTRACT. Let  $r \geq 2$  and  $s \geq 2$  be distinct integers. We establish that, if  $r$  and  $s$  are multiplicatively independent, then the  $r$ -ary expansion and the  $s$ -ary expansion of an irrational real number, viewed as infinite words on  $\{0, 1, \dots, r-1\}$  and  $\{0, 1, \dots, s-1\}$ , respectively, cannot have simultaneously a low block complexity. In particular, they cannot be both Sturmian words.

## 1. INTRODUCTION

Throughout this paper,  $\lfloor x \rfloor$  denotes the greatest integer less than or equal to  $x$  and  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ . Let  $b \geq 2$  be an integer. For a real number  $\xi$ , write

$$\xi = \lfloor \xi \rfloor + \sum_{k \geq 1} \frac{a_k}{b^k} = \lfloor \xi \rfloor + 0.a_1a_2\dots,$$

where each digit  $a_k$  is an integer from  $\{0, 1, \dots, b-1\}$  and infinitely many digits  $a_k$  are not equal to  $b-1$ . The sequence  $\mathbf{a} := (a_k)_{k \geq 1}$  is uniquely determined by the fractional part of  $\xi$ . With a slight abuse of notation, we call it the  $b$ -ary expansion of  $\xi$  and we view it also as the infinite word  $\mathbf{a} = a_1a_2\dots$  over the alphabet  $\{0, 1, \dots, b-1\}$ .

For an infinite word  $\mathbf{x} = x_1x_2\dots$  over a finite alphabet and for a positive integer  $n$ , set

$$p(n, \mathbf{x}) = \text{Card}\{x_{j+1}\dots x_{j+n} : j \geq 0\}.$$

This notion from combinatorics on words is now commonly used to measure the complexity of the  $b$ -ary expansion of a real number  $\xi$ . Indeed, for a positive integer  $n$ , we denote by  $p(n, \xi, b)$  the total number of distinct blocks of  $n$  digits in the  $b$ -ary

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2010 *Mathematics Subject Classification*. 11A63 (primary); 68R15 (secondary).

*Key words and phrases*. Combinatorics on words, Sturmian word, complexity,  $b$ -ary expansion.

expansion  $\mathbf{a}$  of  $\xi$ , that is,

$$p(n, \xi, b) := p(n, \mathbf{a}) = \text{Card}\{a_{j+1} \dots a_{j+n} : j \geq 0\}.$$

Obviously, we have  $1 \leq p(n, \xi, b) \leq b^n$ , and both inequalities are sharp. In particular, the entropy of  $\xi$  to base  $b$ , denoted by  $E(\xi, b)$  and defined by (note that the sequence  $(\log p(n, \xi, b))_{n \geq 1}$  is sub-additive, thus the limit below exists)

$$E(\xi, b) := \lim_{n \rightarrow +\infty} \frac{\log p(n, \xi, b)}{n},$$

satisfies

$$0 \leq E(\xi, b) \leq \log b.$$

If  $\xi$  is rational, then its  $b$ -ary expansion is ultimately periodic and the numbers  $p(n, \xi, b)$ ,  $n \geq 1$ , are uniformly bounded by a constant depending only on  $\xi$  and  $b$ . If  $\xi$  is irrational, then, by a classical result of Morse and Hedlund [9], we know that  $p(n, \xi, b) \geq n + 1$  for every positive integer  $n$ , and this inequality is sharp.

**Definition 1.1.** A Sturmian word  $\mathbf{x}$  is an infinite word which satisfies

$$p(n, \mathbf{x}) = n + 1, \quad \text{for } n \geq 1.$$

A quasi-Sturmian word  $\mathbf{x}$  is an infinite word which satisfies

$$p(n, \mathbf{x}) = n + k, \quad \text{for } n \geq n_0,$$

for some positive integers  $k$  and  $n_0$ .

There exist uncountably many Sturmian words over  $\{0, 1\}$ ; see, e.g., [1].

The following rather general problem was investigated in [4]. Recall that two positive integers  $x$  and  $y$  are called *multiplicatively independent* if the only pair of integers  $(m, n)$  such that  $x^m y^n = 1$  is the pair  $(0, 0)$ .

**Problem 1.2.** *Are there irrational real numbers having a ‘simple’ expansion in two multiplicatively independent bases?*

Among other results, Theorem 2.3 of [4] asserts that any irrational real number cannot have simultaneously too many zeros in its  $r$ -ary expansion and in its  $s$ -ary expansion when  $r$  and  $s$  are multiplicatively independent positive integers. Furthermore, by Theorem 2.1 of [4], there are irrational numbers having maximal

entropy in no base. More precisely, for any real number  $\varepsilon > 0$  and any integer  $b_0 \geq 2$ , there exist uncountably many real numbers  $\xi$  such that

$$E(\xi, b_0) < \varepsilon \quad \text{and} \quad E(\xi, b) < \log b, \quad \text{for } b \geq 2.$$

However, we still do not know whether there exist irrational real numbers having zero entropy in two multiplicatively independent bases. The main purpose of the present work is to make a small step towards the resolution of this problem, by establishing that the complexity function of the  $r$ -ary expansion of an irrational real number and that of its  $s$ -ary expansion cannot both grow too slowly when  $r$  and  $s$  are multiplicatively independent positive integers.

**Theorem 1.3.** *Let  $r$  and  $s$  be multiplicatively independent positive integers. Any irrational real number  $\xi$  satisfies*

$$\lim_{n \rightarrow +\infty} (p(n, \xi, r) + p(n, \xi, s) - 2n) = +\infty.$$

*Said differently,  $\xi$  cannot have simultaneously a quasi-Sturmian  $r$ -ary expansion and a quasi-Sturmian  $s$ -ary expansion.*

Theorem 1.3 answers Problem 3 of [4] and gives the first contribution to Problem 10.21 of [3].

At the heart of the proof of Theorem 1.3 lies the rather surprising fact that we can obtain very precise information on the continued fraction expansion of any real number whose expansion in some integer base is given by a quasi-Sturmian word; see Theorem 3.1 below. This fact was already used in the proof of Theorem 4.5 of [5]. The proof of Theorem 1.3 also depends on the  $S$ -unit theorem, whose proof ultimately rests on the Schmidt Subspace Theorem.

We complement Theorem 1.3 by the following statement addressing expansions of a real number in two multiplicatively dependent bases.

**Theorem 1.4.** *Let  $r, s \geq 2$  be multiplicatively dependent integers and  $m, \ell$  be the smallest positive integers such that  $r^m = s^\ell$ . Then, there exist uncountably many real numbers  $\xi$  satisfying*

$$\lim_{n \rightarrow +\infty} (p(n, \xi, r) + p(n, \xi, s) - 2n) = m + \ell$$

and every irrational real number  $\xi$  satisfies

$$\lim_{n \rightarrow +\infty} (p(n, \xi, r) + p(n, \xi, s) - 2n) \geq m + \ell.$$

The proof of Theorem 1.4 consists in rather tedious combinatorial constructions and is given in [6].

Our paper is organized as follows. Section 2 gathers auxiliary results on Sturmian and quasi-Sturmian words. Theorem 1.3 is proved in Section 3.

## 2. AUXILIARY RESULTS

Our proof makes use of the complexity function studied in [5], which involves the smallest return time of a factor of an infinite word. For an infinite word  $\mathbf{x} = x_1x_2 \dots$  set

$$r(n, \mathbf{x}) = \min\{m \geq 1 : x_i \dots x_{i+n-1} = x_{m-n+1} \dots x_m \text{ for some } 1 \leq i \leq m - n\}.$$

Said differently,  $r(n, \mathbf{x})$  denotes the length of the smallest prefix of  $\mathbf{x}$  containing two (possibly overlapping) occurrences of some word of length  $n$ .

The next lemma gathers several properties of the function  $n \mapsto r(n, \mathbf{x})$  established in [5].

**Lemma 2.1.** *Let  $\mathbf{x}$  be an infinite word.*

(1) *For any positive integer  $n$ , we have*

$$r(n+1, \mathbf{x}) \geq r(n, \mathbf{x}) + 1 \quad \text{and} \quad r(n, \mathbf{x}) \leq p(n, \mathbf{x}) + n.$$

(2) *The word  $\mathbf{x}$  is ultimately periodic if and only if  $r(n+1, \mathbf{x}) = r(n, \mathbf{x}) + 1$  for every sufficiently large  $n$ .*

(3) *If the positive integer  $n$  satisfies  $r(n+1, \mathbf{x}) \geq r(n, \mathbf{x}) + 2$ , then  $r(n+1, \mathbf{x}) \geq 2n+3$ .*

We will make use of the following characterisation of quasi-Sturmian words.

**Lemma 2.2.** *An infinite word  $\mathbf{x}$  written over a finite alphabet  $\mathcal{A}$  is quasi-Sturmian if and only if there are a finite word  $W$ , a Sturmian word  $\mathbf{s}$  defined over  $\{0, 1\}$  and a morphism  $\phi$  from  $\{0, 1\}^*$  into  $\mathcal{A}^*$  such that  $\phi(01) \neq \phi(10)$  and*

$$\mathbf{x} = W\phi(\mathbf{s}).$$

*Proof.* See [7]. □

Throughout this paper, for a finite word  $W$  and an integer  $t$ , we write  $W^t$  for the concatenation of  $t$  copies of  $W$  and  $W^\infty$  for the concatenation of infinitely many copies of  $W$ . We denote by  $|W|$  the length of  $W$ , that is, the number of letters composing  $W$ . A word  $U$  is called periodic if  $U = W^t$  for some finite word  $W$  and an integer  $t \geq 2$ . If  $U$  is periodic, then the period of  $U$  is defined as the length of the shortest word  $W$  for which there exists an integer  $t \geq 2$  such that  $U = W^t$ .

**Lemma 2.3.** *Let  $\mathbf{s}$  be a quasi-Sturmian word and  $W$  be a factor of  $\mathbf{s}$ . Then, there exists a positive integer  $t$  such that the word  $W^t$  is not a factor of  $\mathbf{s}$ .*

This result is certainly well-known. For the sake of completeness, we provide its proof. A factor  $U$  is called a right (resp. left) special word of an infinite word  $\mathbf{x}$  if there are two distinct letters  $a, b$  such that  $Ua, Ub$  (resp.  $aU, bU$ ) are both factors of  $\mathbf{x}$ . If  $V$  occurs infinitely many times in an infinite word  $\mathbf{x}$  which is not ultimately periodic, then there are a right special word  $U_1$  and a left special word  $U_2$  such that  $V$  is a prefix of  $U_1$  and a suffix of  $U_2$ .

*Proof.* Let  $\mathbf{s}$  be a quasi-Sturmian word and  $W$  a factor of  $\mathbf{s}$  such that  $W^t$  is a factor of  $\mathbf{s}$  for any positive integer  $t$ . Since  $n \mapsto p(n, \mathbf{s})$  is strictly increasing [9], there is an integer  $N$  such that  $p(n+1, \mathbf{s}) = p(n, \mathbf{s}) + 1$  for all  $n \geq N$ . Thus, for every  $n \geq N$ , the word  $\mathbf{s}$  has only one right (resp. left) special word of length  $n$ . Since  $\mathbf{s}$  is not ultimately periodic, there exist infinitely many integers  $u$  such that  $VW^uV'$  is a factor of  $\mathbf{s}$  for some  $V, V' \neq W$  with  $|V| = |V'| = |W|$ . Consequently, there exist words  $U, U'$  with  $|U'| > |U| \geq \max(N, |W|)$  and letters  $a, b, c, d, a', b', c', d'$  such that

$$aUb, a'U'b' \text{ are factors of } W^\infty, \quad cUd, c'U'd' \text{ are factors of } \mathbf{s},$$

$$\text{but } a \neq c, b \neq d, a' \neq c', b' \neq d'.$$

Since  $p(n, W^\infty) \leq |W|$  for every positive integer  $n$ , the condition  $|U'| > |U| \geq |W|$  implies that  $cU, c'U', Ud, U'd'$  are not factors of  $W^\infty$ . Furthermore, our assumption that  $W^t$  is a factor of  $\mathbf{s}$  for any positive integer  $t$  implies that the words  $aUb$  and  $a'U'b'$  are factors of  $\mathbf{s}$ . Thus the words  $U$  and  $U'$  are right special and left special words of  $\mathbf{s}$ . Since the right (resp. left) special word of  $\mathbf{s}$  of length  $|U|$  is unique, we

deduce that  $U$  is a prefix and a suffix of  $U'$  and we infer that  $a = a'$ ,  $b = b'$ ,  $c = c'$ ,  $d = d'$ . Therefore,  $cUb$  is a prefix of  $c'U'$  and  $aUd$  is a suffix of  $U'd'$ , which implies that  $cU$  and  $aU$  are two distinct right special words of  $\mathbf{s}$  of length greater than  $N$ , a contradiction.  $\square$

### 3. RATIONAL APPROXIMATION TO QUASI-STURMIAN NUMBERS

To establish Theorem 1.3 we need a precise description of the convergents to quasi-Sturmian numbers. The first assertion (and even a stronger version) of the next theorem has been proved in [5].

**Theorem 3.1.** *Let  $\xi$  be a real number whose  $b$ -ary expansion is a quasi-Sturmian word. There exist infinitely many rational numbers  $\frac{p}{q}$  with  $q \geq 1$  such that*

$$(3.1) \quad \left| \xi - \frac{p}{q} \right| < \frac{1}{q^{5/2}}.$$

*Furthermore, there exists an integer  $M$ , depending only on  $\xi$  and  $b$ , such that, for every reduced rational number  $\frac{p}{q}$  satisfying*

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{Mq^2},$$

*with  $q$  sufficiently large, there exist integers  $r \geq 0$ ,  $s \geq 1$  and  $m$  with  $1 \leq m \leq M$  and  $q = \frac{b^r(b^s - 1)}{m}$ .*

*Proof.* For the proof of the second assertion, we assume that the reader is familiar with the theory of continued fractions (see e.g. Section 1.2 of [2]).

We may assume that  $\lfloor \xi \rfloor = 0$  and write

$$\xi = \sum_{k \geq 1} \frac{x_k}{b^k} = [0; a_1, a_2, \dots].$$

Let  $(\frac{p_j}{q_j})_{j \geq 1}$  denote the sequence of convergents to  $\xi$ .

Since  $\xi$  is irrational, it follows from Lemma 2.1 (2) that the increasing sequence  $\mathcal{N} := (n_k)_{k \geq 1}$  of all the integers  $n$  such that  $r(n+1, \mathbf{x}) \geq r(n, \mathbf{x}) + 2$  is an infinite sequence.

Since  $\mathbf{x}$  is a quasi-Sturmian sequence, there exist integers  $n_0$  and  $\rho$  such that

$$p(n, \mathbf{x}) \leq n + \rho, \quad \text{for } n \geq 1, \text{ with equality for } n \geq n_0.$$

By Lemma 2.1 (1), we deduce that  $r(n, \mathbf{x}) \leq 2n + \rho$  for  $n \geq 1$ .

Let  $k$  be a positive integer. By Lemma 2.1 (3), we have  $r(n_k + 1, \mathbf{x}) \geq 2n_k + 3$ . Define  $\rho_k$  by  $r(n_k + 1, \mathbf{x}) = 2n_k + \rho_k + 1$  and observe that  $2 \leq \rho_k \leq \rho + 1$ . We deduce from the definition of the sequence  $\mathcal{N}$  that

$$(3.2) \quad r(n_k + \ell, \mathbf{x}) = 2n_k + \rho_k + \ell, \quad 1 \leq \ell \leq n_{k+1} - n_k.$$

Set  $\alpha_k = \frac{r(n_k, \mathbf{x})}{n_k}$  and observe that  $\alpha_k \leq 2 + \frac{\rho}{n_k}$ . It follows from the choice of  $n_k$  and Lemma 2.1 (1) that

$$\liminf_{k \rightarrow +\infty} \alpha_k = \liminf_{n \rightarrow +\infty} \frac{r(n, \mathbf{x})}{n},$$

which is shown to be less than 2 in [5]. Consequently, there are infinitely many  $k$  such that  $\alpha_k < 2$ . Let  $k$  be an integer with this property.

By Theorem 1.5.2 of [1], the fact that  $r(n_k, \mathbf{x}) = \alpha_k n_k < 2n_k$  implies that there are finite words  $W_k, U_k, V_k$  of lengths  $w_k, u_k, v_k$ , respectively, and a positive integer  $t$  such that  $W_k(U_k V_k)^{t+1} U_k$  is a prefix of  $\mathbf{x}$  of length  $\alpha_k n_k$  and

$$t(u_k + v_k) + u_k = n_k.$$

Thus, there exists an integer  $r_k$  such that the  $\alpha_k n_k$  first digits of  $\mathbf{x}$  and those of the  $b$ -ary expansion  $W_k(U_k V_k)^\infty$  of the rational number  $\frac{r_k}{b^{w_k}(b^{u_k+v_k}-1)}$  coincide. Consequently, we get

$$\left| \xi - \frac{r_k}{b^{w_k}(b^{u_k+v_k}-1)} \right| \leq \frac{1}{b^{\alpha_k n_k}}.$$

Also, since  $u_k + v_k + w_k = (\alpha_k - 1)n_k$ , we have

$$b^{w_k}(b^{u_k+v_k}-1) \leq b^{(\alpha_k-1)n_k}.$$

A classical theorem of Legendre (see e.g. Theorem 1.8 of [2]) asserts that, if the irrational real number  $\zeta$  and the rational number  $\frac{p}{q}$  with  $q \geq 1$  satisfy  $|\zeta - \frac{p}{q}| \leq \frac{1}{2q^2}$ , then  $\frac{p}{q}$  is a convergent of the continued fraction expansion of  $\zeta$ .

As

$$2(b^{w_k}(b^{u_k+v_k}-1))^2 \leq 2b^{2(\alpha_k-1)n_k} \leq b^{\alpha_k n_k}$$

holds if  $\alpha_k n_k \leq 2n_k - 1$ , Legendre's theorem and the assumption  $\alpha_k < 2$  imply that the rational number  $\frac{r_k}{b^{w_k}(b^{u_k+v_k}-1)}$ , which may not be written under its reduced form, is a convergent, say  $\frac{p_h}{q_h}$ , of the continued fraction expansion of  $\xi$ .

Let  $\ell$  be the smallest positive integer such that  $\alpha_{k+\ell} < 2$ .

We first establish that  $\ell \leq 2$  if  $n_k$  is sufficiently large.

Assume that  $r(n_{k+1}, \mathbf{x}) \geq 2n_{k+1}$  and  $r(n_{k+2}, \mathbf{x}) \geq 2n_{k+2}$ . Since

$$\begin{aligned}
 r(n_{k+2}, \mathbf{x}) &= r(n_{k+1} + (n_{k+2} - n_{k+1}), \mathbf{x}) \\
 (3.3) \quad &= 2n_{k+1} + \rho_{k+1} + n_{k+2} - n_{k+1} \\
 &= n_{k+2} + n_{k+1} + \rho_{k+1}
 \end{aligned}$$

by (3.2), we get  $n_{k+2} - n_{k+1} \leq \rho_{k+1}$ . Put  $\eta_k := r(n_{k+2}, \mathbf{x}) - r(n_{k+1}, \mathbf{x})$ . Then, it follows from (3.3) that

$$\eta_k \leq n_{k+2} + n_{k+1} + \rho_{k+1} - 2n_{k+1} = n_{k+2} - n_{k+1} + \rho_{k+1} \leq 2\rho_{k+1},$$

thus  $\eta_k \in \{1, 2, \dots, 2\rho + 2\}$ .

Since, for any sufficiently large  $n \geq 1$ , we have  $p(n+1, \mathbf{x}) = p(n, \mathbf{x}) + 1$ , there exists a unique factor  $Z_n$  of  $\mathbf{x}$  of length  $n$  such that  $Z_n$  is the prefix of exactly two distinct factors of  $\mathbf{x}$  of length  $n+1$ .

It follows from our assumption  $r(n_{k+1} + 1, \mathbf{x}) > r(n_{k+1}, \mathbf{x}) + 1$  that

$$Z_{n_{k+1}} = x_{r(n_{k+1}, \mathbf{x}) - n_{k+1} + 1} \dots x_{r(n_{k+1}, \mathbf{x})}.$$

Likewise, we get

$$Z_{n_{k+2}} = x_{r(n_{k+2}, \mathbf{x}) - n_{k+2} + 1} \dots x_{r(n_{k+2}, \mathbf{x})}$$

and, since  $Z_{n_{k+1}}$  is a suffix of  $Z_{n_{k+2}}$ , we get

$$\begin{aligned}
 Z_{n_{k+1}} &= x_{r(n_{k+1}, \mathbf{x}) - n_{k+1} + 1} \dots x_{r(n_{k+1}, \mathbf{x})} = x_{r(n_{k+2}, \mathbf{x}) - n_{k+1} + 1} \dots x_{r(n_{k+2}, \mathbf{x})} \\
 &= x_{r(n_{k+1}, \mathbf{x}) + \eta_k - n_{k+1} + 1} \dots x_{r(n_{k+1}, \mathbf{x}) + \eta_k}.
 \end{aligned}$$

It then follows from Theorem 1.5.2 of [1] that there exists an integer  $t_k$ , a word  $T_k$  of length  $\eta_k$  and a prefix  $T'_k$  of  $T_k$  such that

$$Z_{n_{k+1}} = (T_k)^{t_k} T'_k.$$

We deduce that

$$t_k \geq \frac{n_{k+1} - \eta_k + 1}{\eta_k} \geq \frac{n_{k+1} - 2\rho - 1}{2(\rho + 1)}.$$

By Lemma 2.3, there exists an integer  $t$  such that, for every factor  $W$  of  $\mathbf{x}$  of length at most  $2\rho + 2$ , the word  $W^t$  is not a factor of  $\mathbf{x}$ . We conclude that  $n_{k+1}$ , hence  $k$ , must be bounded.

Consequently, if  $k$  is sufficiently large, then we cannot have simultaneously  $r(n_{k+1}, \mathbf{x}) \geq 2n_{k+1}$  and  $r(n_{k+2}, \mathbf{x}) \geq 2n_{k+2}$ . This implies that  $\ell = 1$  or  $\ell = 2$ .



Since  $\alpha_{k+\ell} < 2$ , it follows from Legendre's theorem that the rational number  $\frac{r_{k+\ell}}{b^{w_{k+\ell}}(b^{u_{k+\ell}+v_{k+\ell}-1})}$ , defined analogously as the rational number  $\frac{r_k}{b^{w_k}(b^{u_k+v_k-1})}$  and which may not be written under its reduced form, is a convergent, say  $\frac{p_j}{q_j}$ , of the continued fraction expansion of  $\xi$ .

Here, the indices  $h$  and  $j$  depend on  $k$ . We have

$$q_h \leq b^{w_k}(b^{u_k+v_k} - 1) \leq b^{(\alpha_k-1)n_k}, \quad q_j \leq b^{w_{k+\ell}}(b^{u_{k+\ell}+v_{k+\ell}} - 1) \leq b^{(\alpha_{k+\ell}-1)n_{k+\ell}}.$$

Note that it follows from (3.3) that

$$(\alpha_{k+2} - 1)n_{k+2} = r(n_{k+2}, \mathbf{x}) - n_{k+2} \leq n_{k+1} + \rho + 1,$$

and, likewise,

$$(\alpha_{k+1} - 1)n_{k+1} \leq n_k + \rho + 1.$$

In particular, we have  $n_{k+1} \leq n_k + \rho + 1$  if  $\alpha_{k+1} \geq 2$ .

The properties of continued fractions give that

$$\frac{1}{2q_h q_{h+1}} < \left| \xi - \frac{p_h}{q_h} \right| < \frac{1}{q_h q_{h+1}}, \quad \frac{1}{2q_j q_{j+1}} < \left| \xi - \frac{p_j}{q_j} \right| < \frac{1}{q_j q_{j+1}}.$$

This implies that

$$q_{j+1} > \frac{b^{\alpha_{k+\ell} n_{k+\ell}}}{2q_j} \geq \frac{b^{n_{k+\ell}}}{2}.$$

Since  $\alpha_k < 2$ , we get

$$q_h \leq b^{(\alpha_k-1)n_k} < b^{n_k} \leq \frac{b^{n_{k+\ell}}}{2} < q_{j+1}.$$

Combined with  $\frac{p_h}{q_h} \neq \frac{p_j}{q_j}$ , this gives

$$q_h < q_{h+1} \leq q_j < q_{j+1}.$$

It follows from

$$q_h > \frac{b^{\alpha_k n_k}}{2q_{h+1}}$$

and

$$q_{h+1} \leq q_j \leq b^{(\alpha_{k+\ell}-1)n_{k+\ell}} \leq b^{n_{k+\ell}+2(\rho+1)},$$

that

$$q_h > \frac{b^{\alpha_k n_k}}{2b^{n_{k+\ell}+2(\rho+1)}} = \frac{b^{(\alpha_k-1)n_k}}{2b^{2(\rho+1)}}.$$

Since  $q_h \leq b^{w_k}(b^{u_k+v_k} - 1) \leq b^{(\alpha_k-1)n_k}$ , this shows that the rational number  $\frac{r_k}{b^{w_k}(b^{u_k+v_k-1})}$  is not far from being reduced, in the sense that the greatest common

divisor of its numerator and denominator is at most equal to  $2b^{2(\rho+1)}$ . Furthermore, it follows from

$$q_{h+1} > \frac{b^{\alpha_k n_k}}{2q_h} \geq \frac{b^{n_k}}{2}$$

that

$$1 \leq \frac{q_j}{q_{h+1}} \leq 2b^{2(\rho+1)}.$$

Consequently, all the partial quotients  $a_{h+2}, \dots, a_j$  are less than  $2b^{2(\rho+1)}$  and we get

$$\left| \xi - \frac{p_\ell}{q_\ell} \right| > \frac{1}{(a_{\ell+1} + 2)q_\ell^2} \geq \frac{1}{2(b^{2(\rho+1)} + 1)q_\ell^2}, \quad \text{for } \ell = h+1, \dots, j-1.$$

Consequently, the second assertion of the theorem holds with the value  $M = 2(b^{2(\rho+1)} + 1)$ .  $\square$

*Proof of Theorem 1.3.*

Let  $\xi$  be a real number whose  $r$ -ary expansion and whose  $s$ -ary expansions are quasi-Sturmian words. Since (3.1) has infinitely many solutions and by unicity of the continued fraction expansion of an irrational real number, we deduce from Theorem 3.1 that there are positive integers  $m_1, m_2$  and infinitely many quadruples  $(u_1, v_1, u_2, v_2)$  of non-negative integers with  $v_1 v_2 \neq 0$  such that

$$\frac{r^{u_1}(r^{v_1} - 1)}{m_1} = \frac{s^{u_2}(s^{v_2} - 1)}{m_2}.$$

In particular, the equation

$$(3.4) \quad \frac{m_2}{m_1} r^{z_1} s^{-z_4} - \frac{m_2}{m_1} r^{z_2} s^{-z_4} + s^{z_3} = 1$$

has infinitely many solutions  $(z_1, \dots, z_4)$  in non-negative integers with  $z_3 \geq 1$ .

This is a linear equation in variables which lie in a multiplicative group generated by  $r$  and  $s$ . By Theorem 1.1 of [8], Equation (3.4) has only finitely many non-degenerate solutions. Since  $r$  and  $s$  are multiplicatively independent, the equation  $\frac{m_2}{m_1} r^{z_2} s^{-z_4} = s^{z_3}$  has only finitely many solutions. Consequently, Equation (3.4) has only finitely many solutions, a contradiction to our assumption. Consequently, the  $r$ -ary and the  $s$ -ary expansions of  $\xi$  cannot both be quasi-Sturmian.

## ACKNOWLEDGEMENT

Dong Han Kim was supported by the National Research Foundation of Korea (NRF-2015R1A2A2A01007090). The authors are grateful to the referee for a very careful reading.

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